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Numerical simulations of asymmetric first-price auctions

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ABSTRACT

The standard method for computing the equilibrium strategies of asymmetric first-price auctions is the backward-shooting method. In this study we show that the backward-shooting method is inherently unstable, and that this instability cannot be eliminated by changing the numerical methodology of the backward solver. Moreover, this instability becomes more severe as the number of players increases. We then present a novel boundary-value method for computing the equilibrium strategies of asymmetric first-price auctions. We demonstrate the robustness and stability of this method for auctions with any number of players, and for players with mixed types of distributions, including distributions with more than one crossing. Finally, we use the boundary-value method to study large auctions with hundreds of players, to compute the asymptotic rate at which large first-price and second-price auctions become revenue equivalent, and to study auctions in which the distributions cannot be ordered according to first-order stochastic dominance.

1. Introduction

The theory of auctions has mostly dealt with symmetric auctions, in which the valuations of all bidders are drawn according to the same distribution function. In practice, however, it often happens that bidders are asymmetric, i.e., their valuations are drawn according to different distribution functions. In such cases, the mathematical model is considerably more complex. As a result, relatively little is known at present on asymmetric auctions.

Numerical simulations can play an important role in the study of asymmetric auctions. A pioneering contribution was made by Marshall et al. (1994), who developed the first numerical method capable of computing the equilibrium bids in asymmetric first-price auctions, which is based on backward shooting. The backward-shooting method was used in numerous subsequent studies (e.g., Fibich and Gavious, 2003; Fibich et al., 2002, 2004; Gayle and Richard, 2008; Li and Riley, 2007; Maskin and Riley, 2000), and has become the standard method for computing the equilibrium bids in asymmetric first-price auctions. Nevertheless, the backward-shooting method is far from optimal. Indeed, Marshall et al. (1994) observed that "backward solutions are well behaved except in neighborhoods of the origin where they become (highly) unstable". Similarly, Li and Riley (1997) noted that "because the differential equations which define the equilibrium bid functions are very poorly behaved at the lower end-point, there are some quite complex technical issues which had to be dealt with before we could develop such a program."

In this work we show that the instability of the backward-shooting method is not a "technical issue", but rather an inherent analytic property of backward solutions. Therefore, it cannot be eliminated by changing the numerical methodology for the backward integration. Moreover, this inherent instability becomes more severe as the number of players increases.

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Therefore, a robust numerical solver of the equilibrium bids in asymmetric first-price auctions requires a completely different approach.

In this study we develop a novel method, the boundary-value method, for computing the equilibrium bids in asymmetric first-price auctions. In this method, the system of nonlinear ordinary differential equations for the equilibrium strategies is solved as a boundary-value problem, rather than with backward shooting. A priori, this approach is problematic, since the location of the right boundary is unknown. However, we eliminate this problem by changing the independent variable, so that the transformed system of ordinary differential equations resides on a fixed known domain. Therefore, the transformed nonlinear system can be solved using standard methods, such as fixed-point iterations or Newton's iterations.

Our simulations show that the boundary-value method is considerably more robust than backward shooting. In particular, unlike backward shooting, it performs well even when the number of players is large, and for mixed types of distributions, including all common distributions (Weibull, truncated normal, beta distributions, etc.), as well as distributions with more than one crossing. We note, however, that the evidence for the performance of the boundary-value method is purely numerical. In particular, we do not prove that the iterative solution exists and is well defined, or that the iterations converge.

The paper is organized as follows. In Section 2 we present the model of asymmetric first-price auctions. In Section 3 we show that the backward-shooting method is inherently unstable, and that this instability increases with the number of players. The new boundary-value method is presented in Section 4 for two players, and in Section 5 for any number of players.

In Section 6 we use the boundary-value method to study several problems which could not be studied numerically using backward shooting.

1. In Section 6.1 we take advantage of the fact that the boundary-value method does not become unstable as the number of players increases, and study large auctions *with as many as* 450 *players and with as many as nine different types of players*.
2. In Section 6.2 we study the revenue difference between large first-price and second-price auctions. Bali and Jackson (2002) showed that this revenue difference goes to zero as the number of players goes to infinity. That study, however, did not consider *the rate* at which first-price and second-price auctions become revenue equivalent. Using the boundary-value method, we show numerically that *the revenue difference between the first-price and second-price auction decays as* $1/n^3$ *or even faster*, where n is the number of players. This numerical observation suggests that the problem of revenue ranking in large asymmetric auctions may be more of academic interest than of practical value.
3. In Section 6.3 we take advantage of the fact that the boundary-value method is not restricted to certain distribution functions. We first relax the common assumption of stochastic dominance, and show numerically that analytic results that were proved under this assumption do not necessarily extend to the general case. We then use the boundary-value method to provide the first numerical example of bidding strategies which cross each other more than once.

Final remarks are given in Section 7. Matlab codes are provided in the on-line supplement.

2. Model formulation

Consider n risk-neutral players bidding for a single object in a first-price auction. The value v_i of the object for the ith player ($i = 1, \dots, n$) is private information to i, and is drawn independently from the interval $[0, 1]$ according to a distribution function $F_i(v)$, such that $F_i(0) = 0$ and $F_i(1) = 1$. The functions $\{F_i(v)\}$ are common knowledge and have a continuous density $f_i(v) = F_i'(v) > 0$ in $(0, 1]$. Each bidder submits a sealed bid of $b_i = b_i(v_i)$. The highest bidder wins the object and pays his bid. All other bidders do not pay anything.

We denote by $v_i(b_i)$ the inverse of the equilibrium bidding strategy $b_i(v_i)$. Then, the equations for $\{v_i\}_{i=1}^n$ are given by Maskin and Riley (2000):

$$v_i'(b) = \frac{F_i(v_i(b))}{f_i(v_i(b))}\left[\left(\frac{1}{n-1}\sum_{j=1}^{n}\frac{1}{v_j(b)-b}\right) - \frac{1}{v_i(b)-b}\right], \quad i = 1, \dots, n. \tag{1a}$$

The initial conditions for the system (1a) are

$$v_i(0) = 0, \quad i = 1, \dots, n. \tag{1b}$$

The initial value problem (1a), (1b) does not have a unique solution. Indeed, one cannot apply the standard local existence and uniqueness theorem, since the right-hand side of (1), (1b) is of the form $\frac{0}{0}$. The equilibrium strategies, however, satisfy the additional condition that the maximal bid of all bidders is the same (Maskin and Riley, 2000). In other words, there exists some $0 < \bar{b} < 1$, such that

$$v_i(\bar{b}) = 1, \quad i = 1, \dots, n. \tag{1c}$$

The existence and uniqueness of solutions to (1a), (1b), (1c) was proved by Lebrun (1996, 1999, 2006).

3. Instability of the backward-shooting method

3.1. The backward-shooting method

The ODE system (1) is a nonlinear boundary-value problem, in which the location of the right boundary $\bar{b}$ is unknown. Therefore, from a numerical point of view, it is natural to solve (1) using a forward-marching approach, i.e., search for the solution of the initial value problem (1a), (1b) that satisfies (1c). Since the right-hand side of (1a) at $b=0$ is of the form $\frac{0}{0}$, a special treatment is required at $b=0$ in order to solve (1a), (1b) numerically. For example, Marshall et al. (1994) obtained an analytic approximation $v_i(b=h) \approx v_{i,h}$ of the solution at $0 < h \ll 1$, and then solved (1a) for $b \geqslant h$ with the initial condition

$$v_i(b=h) = v_{i,h}. \tag{2}$$

Marshall et al. observed, however, that the resulting solution of (1a), (1b) does not satisfy the boundary condition (1c). Therefore, they decided to solve the boundary-value problem (1) using a backward-shooting approach. In this approach, one finds the value of $\bar{b}$ by solving Eq. (1a) backwards in b for $b \leqslant \bar{b}_\varepsilon$, subject to the "initial" conditions

$$v_i(\bar{b}_\varepsilon) = 1, \quad i = 1, \ldots, n, \tag{3}$$

and searching for the value of $\bar{b}_\varepsilon$ for which $v_i(0) = 0$ for $i = 1, \ldots, n$.

The backward-shooting method was used to solve Eq. (1) by Marshall et al. (1994), as well as in numerous subsequent studies (e.g., Fibich and Gavious, 2003; Fibich et al., 2002, 2004; Gayle and Richard, 2008; Li and Riley, 2007; Maskin and Riley, 2000), and it is currently the standard method for computing the equilibrium bids in asymmetric first-price auctions. Various variants of the backward-shooting method have been proposed in the literature. Of particular interest is the program BIDCOMP[2] by Li and Riley (2007), which is freely available on-line. This implementation uses an adaptive step size for the numerical backward integration to allow better control of the error. More recently, Gayle and Richard (2008) implemented the backward-shooting method using local Taylor series expansions of the solution, as well as of the distributions $\{F_i\}$.

Although backward shooting is the standard approach for solving (1), it is far from optimal. Indeed, as noted in the Introduction, this method becomes unstable near the left boundary (Marshall et al., 1994; Li and Riley, 2007). In what follows, we analyze the instability of the backward-shooting method in the case when all players are symmetric. This case is much easier to analyze, since the system of Eqs. (1) reduces to a single ordinary differential equation, which can be solved explicitly. As we shall see, this simple model captures all the instability characteristics of backward solutions in the asymmetric case.

3.2. The symmetric model

Let us consider the symmetric case when $F_i(v) \equiv F(v)$ for $i = 1, \ldots, n$, and all bidders have the same equilibrium strategy $b_i(v) = b(v)$. In this case, the system (1) reduces to

$$v'(b) = \frac{1}{n-1} \frac{F(v(b))}{f(v(b))} \frac{1}{v(b) - b}, \tag{4a}$$

with the initial condition

$$v(0) = 0. \tag{4b}$$

Eq. (4) can be solved explicitly as follows. The equation for $b = v^{-1}$ is given by

$$b'(v) = (n-1) \frac{f(v)}{F(v)} \bigl(v - b(v)\bigr), \tag{5a}$$

with the initial condition

$$b(0) = 0. \tag{5b}$$

In order to solve this first-order ordinary differential equation we multiply Eq. (5a) by $e^{\int^v (n-1) \frac{f(s)}{F(s)} ds} = F^{n-1}(v)$, which gives

$$\bigl[F^{n-1}(v) b(v)\bigr]' = \bigl[F^{n-1}(v)\bigr]' v. \tag{6}$$

Integration of this equation between 0 and v, and using the initial condition (5b) gives the explicit solution

$$b(v) = v - \frac{\int_0^v F^{n-1}(s)\, ds}{F^{n-1}(v)}. \tag{7}$$

Therefore, in particular, the maximal bid in a symmetric first-price auction is given explicitly by

$$\bar{b} = b(1) = 1 - \int_0^1 F^{n-1}(s)\,ds. \tag{8}$$

3.3. *Instability of backward solutions of symmetric auctions*

Although the solution of (4) is known, see Eq. (7), in order to analyze the backward-shooting method for Eq. (1), we now "solve" Eq. (4) using backward shooting. Let $v_\varepsilon(b)$ be the solution of (4a) for $b \leqslant \bar{b}_\varepsilon$, with the "initial" condition

$$v_\varepsilon(\bar{b}_\varepsilon) = 1, \quad \bar{b}_\varepsilon = \bar{b} + \varepsilon, \tag{9}$$

where $\bar{b}$ is given by (8). The inverse function $b_\varepsilon = v_\varepsilon^{-1}$ is the solution of Eq. (5a) for $v \leqslant 1$, with the "initial" condition

$$b_\varepsilon(1) = \bar{b}_\varepsilon, \quad \bar{b}_\varepsilon = \bar{b} + \varepsilon. \tag{10}$$

The solution of (5a), (10) can be calculated explicitly:

Lemma 3.1. *Let $b_\varepsilon(v)$ be a solution of* (5a) *for $v \leqslant 1$ with the initial condition* (10)*. Then,*

$$b_\varepsilon(v) = b(v) + \frac{\varepsilon}{F^{n-1}(v)}, \tag{11}$$

where $b(v)$, the solution of (5)*, is given by* (7)*.*

Proof. Integration of (6) between 1 and v and using $b(1) = \bar{b}$ gives

$$b(v) = \frac{\bar{b} + 1 - vF^{n-1}(v) - \int_v^1 F^{n-1}(s)\,ds}{F^{n-1}(v)}. \tag{12}$$

Similarly,

$$b_\varepsilon(v) = \frac{\bar{b} + \varepsilon + 1 - vF^{n-1}(v) - \int_v^1 F^{n-1}(s)\,ds}{F^{n-1}(v)}. \tag{13}$$

Substraction of (13) from (12) proves the result. □

Lemma 3.1 shows that the error $|b_\varepsilon(v) - b(v)|$ increases monotonically as v decreases from $|b_\varepsilon(1) - b(1)| = \varepsilon$ to $\lim_{v\to 0} |b_\varepsilon(v) - b(v)| = \infty$. We now show that there are two types of backward solutions. Indeed, by (7) and (11),

$$b'_\varepsilon(v) = (n-1)\frac{(\int_0^v F^{n-1}(s)\,ds - \varepsilon) f(v)}{F^n(v)}.$$

Therefore,

1. When $\varepsilon < 0$, $b'_\varepsilon(v) > 0$ for $0 < v \leqslant 1$. Hence, as v goes down from 1 to 0, $b_\varepsilon(v)$ decreases monotonically from $\bar{b}_\varepsilon$ to $-\infty$, see Fig. 1A. In particular, there exists a point $0 < v_{min} < 1$ such that $b_\varepsilon(v_{min}) = 0$. Following Lebrun (1996, 1999), this solution is called a *type*-I *backward solution*.
2. When $\varepsilon > 0$, $b'_\varepsilon(v) = 0$ when $\int_0^v F^{n-1}(s)\,ds = \varepsilon$. Therefore, there exists a point $0 < v_{min} < 1$ such that $b'_\varepsilon(v_{min}) = 0$. Hence, by Eq. (5a),

$$b_\varepsilon(v_{min}) = v_{min}. \tag{14}$$

   Thus, as v goes down from 1 to 0, $b_\varepsilon(v)$ decreases from $b_\varepsilon(1) = \bar{b}_\varepsilon$ to $b_\varepsilon(v_{min}) = v_{min}$, and then increases to $+\infty$ as $v \to 0$, see Fig. 1B. Following Lebrun (1996, 1999), this solution is called a *type*-II *backward solution*.

The solution of (5) is bounded by $0 \leqslant b(v) \leqslant v$. Indeed, the bids in first-price auctions (symmetric, as well as asymmetric) have to be nonnegative and below the bidder's value. The maximal interval of $b_\varepsilon(v)$ over which

$$0 \leqslant b_\varepsilon(v) \leqslant v \tag{15}$$

is $[v_{min}, 1]$. Indeed, for $0 \leqslant v < v_{min}$, type-I backward solutions become negative, and type-II backward solutions satisfy $b_\varepsilon(v) > v$.

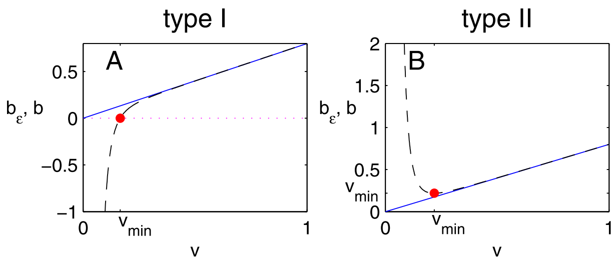

Fig. 1. Analytic solution $b(v)$ of (5) (solid curve) and analytic solution $b_\varepsilon(v)$ of (5a), (10) (dashed curve) with $n = 5$ players and $F = v$. A: $\varepsilon = -0.005$ (type-I). Thick dot denotes the value at which $b_\varepsilon(v_{min}) = 0$. B: $\varepsilon = 0.005$ (type-II). Thick dot denotes the value at which $b'_\varepsilon(v_{min}) = 0$.

Lemma 3.2. *Let $\varepsilon \neq 0$, let $b_\varepsilon(v)$ be a solution of* (5a), (10) *for $v \leqslant 1$, and let $[v_{min}, 1]$ be the maximal interval of $b_\varepsilon(v)$ on which* (15) *holds. Then,*

$$v_{min} \geqslant F^{-1}\left(\sqrt[n-1]{|\varepsilon|}\right) > 0. \tag{16}$$

Proof. When $\varepsilon < 0$, the condition $b_\varepsilon(v) \geqslant 0$ implies that

$$b_\varepsilon(v) = b(v) - \frac{|\varepsilon|}{F^{n-1}(v)} \geqslant 0.$$

Hence,

$$|\varepsilon| \leqslant b(v_{min}) F^{n-1}(v_{min}).$$

Since $b(v_{min}) \leqslant v_{min} \leqslant 1$, it follows that $|\varepsilon| \leqslant F^{n-1}(v_{min})$. Hence, $v_{min} > F^{-1}(\sqrt[n-1]{|\varepsilon|})$.

When $\varepsilon > 0$, the condition $b_\varepsilon(v) \leqslant v$ implies that

$$b_\varepsilon(v) = b(v) + \frac{\varepsilon}{F^{n-1}(v)} = v + \frac{\varepsilon - \int_0^v F^{n-1}(s)\,ds}{F^{n-1}(v)} \leqslant v.$$

Hence, $\varepsilon \leqslant \int_0^{v_{min}} F^{n-1}(s)\,ds$. Since $F(v)$ increases monotonically, it follows that

$$\int_0^{v_{min}} F^{n-1}(s)\,ds < v_{min} F^{n-1}(v_{min}) < F^{n-1}(v_{min}).$$

Hence, $F^{n-1}(v_{min}) > \varepsilon$ and $v_{min} > F^{-1}(\sqrt[n-1]{\varepsilon})$. □

From Lemma 3.2 it follows that the interval $[v_{min}, 1]$ shrinks to zero as $n \to \infty$:

Corrolary 3.1. *Under the conditions of Lemma* 3.2, $\lim_{n\to\infty} v_{min} = 1$.

Proof. This follows immediately from (16). □

Lemma 3.2 and Corollary 3.1 show that

Corrolary 3.2. *Backward solutions become more sensitive to the initial error $\bar{b}_\varepsilon - \bar{b}$ as the number of players increases.*

To illustrate this point, we set $\varepsilon = \pm 10^{-15}$, and solve (5a), (10) with $F = v$ for $n = 10$, 20, and 100 players:

1. When $\varepsilon = -10^{-15}$ then $b(v_{min}) = 0$ (type-I backward solution) and v_{min} increases from 0.033 for $n = 10$ to $v_{min} = 0.17$ for $n = 20$ players to $v_{min} = 0.70$ for $n = 100$ players, see Fig. 2.
2. When $\varepsilon = +10^{-15}$ then $b'_\varepsilon(v_{min}) = 0$ (type-II backward solution). In this case v_{min} increases from 0.039 for $n = 10$ to $v_{min} = 0.20$ for $n = 20$ players to $v_{min} = 0.74$ for $n = 100$ players, see Fig. 3.

Since the error in $\bar{b}$ is of the order of the machine roundoff error, we conclude that *even in the symmetric case, it is impossible to solve* (1) *using backward shooting when the number of players is large* (where $n = 20$ is already "large"). Note that in Figs. 2 and 3 we plotted $b_\varepsilon(v)$ by using the explicit expression (6). Therefore, the backward-shooting error is analytic, i.e., a property of the ordinary differential equation solution, rather than of the numerical solution.

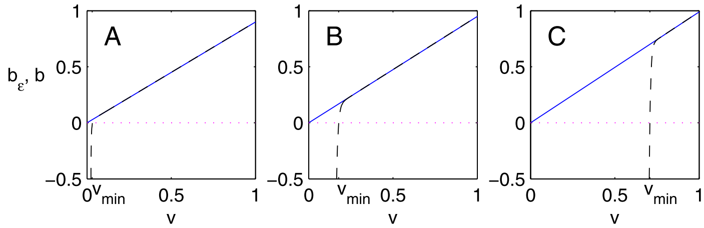

Fig. 2. Solution $b(v)$ of (5) with $F = v$ (solid curve) analytic and solution $b_\varepsilon(v)$ of (5a), (10) with $F = v$ and $\varepsilon = -10^{-15}$ (dashed curve). Dotted curve is the line $b \equiv 0$. Here, v_{min} denotes the value at which $b_\varepsilon(v_{min}) = 0$. A: $n = 10$. B: $n = 20$. C: $n = 100$.

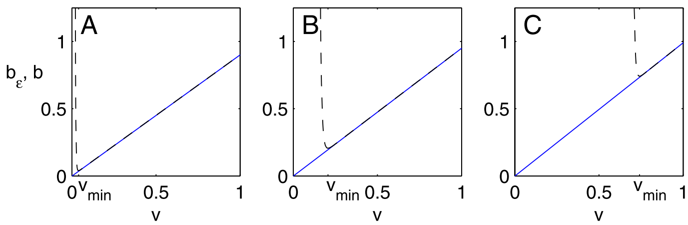

Fig. 3. Same as Fig. 2 with $\varepsilon = +10^{-15}$. v_{min} denotes the value at which $b'_\varepsilon(v_{min}) = 0$ and $b_\varepsilon(v_{min}) = v_{min}$.

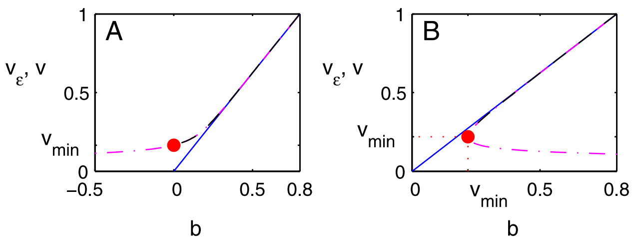

Fig. 4. Analytic solution $v(b)$ of (4) (solid curve) and analytic solution $v_\varepsilon(b)$ of (4a), (9) (dashed curve) with $n = 5$ players and $F = v$. A: $\varepsilon = -0.005$ (type-I). B: $\varepsilon = 0.005$ (type-II). The additional dash-dotted curve in A and B is $v_\varepsilon = b_\varepsilon^{-1}$, where b_ε is taken from Figs. 1A and 1B, respectively.

In the asymmetric case, the equilibrium bid functions are computed by solving (1) for the *inverse* bid functions. Therefore, in analogy with the asymmetric case, we now consider the *analytic error* of $|v_\varepsilon - v|$ when we solve (4a), (9) for v_ε backwards from $b = \bar{b}_\varepsilon$ down to $b = 0$. Since the error $|b_\varepsilon - b|$ increases monotonically as v decreases, the error $|v_\varepsilon - v|$ also increases monotonically as b decreases from $b = \bar{b}$ towards $b = 0$. In Fig. 4 we plot v_ε and v for the same data as in Fig. 1, i.e., $n = 5$, $F = v$ and $\varepsilon = \pm 0.005$:

1. As b decreases from $b = \bar{b}$ to $b = 0$, the type-I backward solution $v_\varepsilon(b)$ goes down from $v_\varepsilon(\bar{b}_\varepsilon) = 1$ to $v_\varepsilon(0) = v_{min} \approx 0.16$, see Fig. 4A. Comparison of v_ε from Fig. 4A with $v_\varepsilon = b_\varepsilon^{-1}$, where b_ε is taken from Fig. 1A shows that the two solutions are identical for $v_{min} \leqslant b \leqslant \bar{b}$. For $0 \leqslant b < v_{min}$, however, b_ε^{-1} becomes negative, but v_ε from Fig. 4A is not defined, since it was computed only for $0 \leqslant b \leqslant b_\varepsilon$.[1]
2. The type-II backward solution $v_\varepsilon(b)$ is defined for $v_{min} \leqslant b \leqslant \bar{b}$, but not for $0 \leqslant b \leqslant v_{min}$, see Fig. 4B. A comparison of v_ε from Fig. 4B with $v_\varepsilon = b_\varepsilon^{-1}$, where b_ε is taken from Fig. 1B, reveals why v_ε is not defined for $0 \leqslant b < v_{min}$. Indeed, as b decreases from $\bar{b}$ to v_{min}, $v_\varepsilon = b_\varepsilon^{-1}$ decreases along the upper branch. Then, at $b = v_{min}$, the trend reverses and b *increases* as $v_\varepsilon = b_\varepsilon^{-1}$ decreases along the lower branch from v_{min} to 0. Therefore, $v_\varepsilon(b)$ is defined only for $b \geqslant v_{min}$.

So far we only considered the analytic error of backward solutions. We now consider an additional numerical error that may occur for type-II inverse solutions. As noted, these solutions are defined only for $v_{min} \leqslant b \leqslant \bar{b}_\varepsilon$. At $b = v_{min}$, $v'_\varepsilon(b)$ becomes infinite. However, depending on the way the numerical ordinary differential equation solver handles the infinite derivative of v_ε at $b = v_{min}$, $v_\varepsilon(b)$ may be (incorrectly) computed for $b < v_{min}$. For example, in Fig. 5A we solve (4a), (9) with $\varepsilon = 0.005$ using Matlab's ODE45 subroutine.[2] In this case, the computed $v_\varepsilon(b)$ agrees with the analytic solution in the region $v_{min} \leqslant b \leqslant \bar{b}$. However, ODE45 does not stop at $b = v_{min}$ (as it should, see Fig. 4B), but rather continues into the

[1] Of course, if we continue to solve (4a), (9) for $b < 0$, then the two solutions would also agree in this region.
[2] In what follows, we use the default parameters of Matlab's solvers.

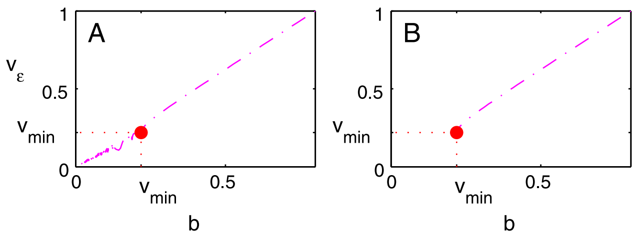

Fig. 5. Numerical solution $v_\varepsilon(b)$ of (4a), (9) with $n = 5$ players, $F = v$, and $\varepsilon = 0.005$ (type-II), computed using A: Matlab's ODE45 subroutine. B: Matlab's ODE15s subroutine.

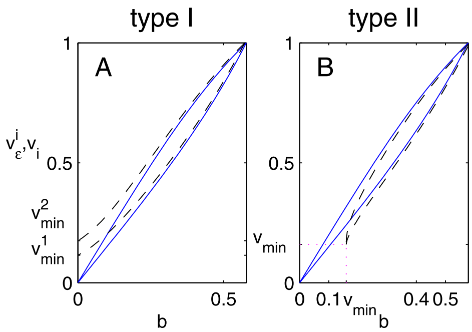

Fig. 6. Backward solution $\mathbf{v}^i_\varepsilon(b)$ of (1a) with the initial condition (17) (dashed curve), and the solution $\mathbf{v}(b)$ of (1) (solid curve) for $n = 2$, $F_1 = v$ and $F_2 = v^2$. A: $\varepsilon = -0.005$. B: $\varepsilon = 0.005$. Note the similarity to Figs. 4A and 4B. C: Error (18) of solution in A. D: Error (18) of solution in B.

region $b < v_{min}$, where the computed values are a numerical artifact. In Fig. 5B we solve the same problem using Matlab's ODE15s subroutine. In this case, the numerical integration does terminate at $b = v_{min}$.[3]

3.4. Instability of backward solutions of asymmetric auctions

In Section 3.3 we saw that the backward-shooting method for solving symmetric auctions is unstable, and that the instability increases with $(\bar{b} - b)$ and with n. Moreover, in the case of type-II solutions we saw that an additional numerical error may occur due to the infinite derivative of the solution at v_{min}. We now show numerically that the same instability characteristics exist in asymmetric auctions.

Let us consider backward solutions of (1a) for $b \leqslant \bar{b}_\varepsilon$ with the "initial" condition

$$v^i_\varepsilon(\bar{b}_\varepsilon) = 1, \quad \bar{b}_\varepsilon = \bar{b} + \varepsilon. \tag{17}$$

Lebrun (1996, 1999) proved that asymmetric backward solutions can be classified into type-I and type-II solutions, such that

1. When $\varepsilon < 0$ (type-I), for each solution $v^i_\varepsilon(0) > 0$. We denote the value of the solution at $b = 0$ as $v^i_{min} := v^i_\varepsilon(0)$. Note that it is possible that $v^i_{min} \neq v^j_{min}$.
2. When $\varepsilon > 0$ (type-II), for each solution $v^i_\varepsilon(b)$ there exists a point $v^i_{min} > 0$ such that $v_\varepsilon(v^i_{min}) = v^i_{min}$. In this case, v^i_{min} is equal for all players, except possibly for one player.

This classification can be seen numerically in Figs. 6A and 6B, where we solve (1a) for $b \leqslant \bar{b}_\varepsilon$ with $n = 2$ players with $F_i = \{v, v^2\}$, and the "initial" condition (17) with $\varepsilon = \pm 0.005$. Note the similarity to Figs. 4A and 4B of the symmetric case.

In the symmetric case we showed that the error $|v_\varepsilon(b) - v(b)|$ increases monotonically in $\bar{b} - b$. While such a result was not proved for asymmetric auctions, Figs. 6C and 6D show numerically that the error

$$\|\mathbf{v}_\varepsilon(b) - \mathbf{v}(b)\| = \sqrt{\sum_{i=1}^{n} \left(v^i_\varepsilon(b) - v_i(b)\right)^2} \tag{18}$$

of backward solutions in the asymmetric case also increases monotonically in $\bar{b} - b$.

[3] ODE15s does not always stop at v_{min}, see Fig. 8B below.

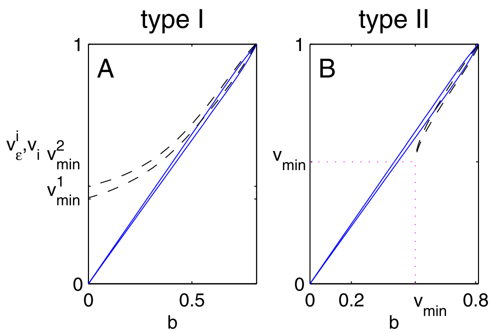

Fig. 7. Same as Fig. 6 for $n = 4$ players with $F_i = \{v, v, v^2, v^2\}$.

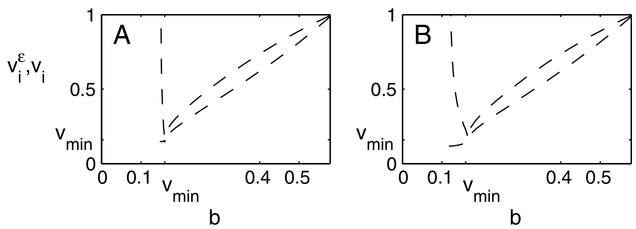

Fig. 8. Numerical solution $v_\varepsilon(b)$ of (4a), (9) with $n = 5$ players, $F = v$, and $\varepsilon = 0.005$ (type-II) computed using A: Matlab's ODE45 subroutine. B: Matlab's ODE15s subroutine. Note the similarity to Fig. 5.

In the symmetric case we showed that the instability of backward solutions increases with n. We now numerical results that suggests that *our conclusion that the instability of backward solutions increases with n, remains valid in the asymmetric case.* To see that, in Fig. 7 we solve (1a) with $n = 4$ players with $F_i = \{v, v, v^2, v^2\}$ for $b \leqslant \bar{b}_\varepsilon$ with the "initial" condition' (17), and observe that v^i_{min} is considerably larger than in the case of two players.[4] Indeed, $\max_i v^i_{min} < 0.2$ for $n = 2$, while $\min_i v^i_{min} > 0.35$ for $n = 4$.

In conclusion, the instability of the backward solutions of asymmetric auctions is essentially the same as for symmetric auctions. The source of the instability is analytic, hence independent of the numerical method used to integrate (1a) backwards. In particular, it is not possible to overcome the instability by changing the numerical method for the backward integration. Moreover, the instability increases as b decreases and as the number of players increases.

As in the symmetric case, an additional numerical error may occur for type-II inverse functions $v_\varepsilon(b)$. These solutions are defined only for $v_{min} \leqslant b \leqslant \bar{b}_\varepsilon$. At $b = v_{min}$, the derivatives dv^i_ε/db become infinite. However, depending on the way numerical ordinary differential equation solvers handle infinite derivatives, $v^i_\varepsilon(b)$ may be (incorrectly) computed for $b < v_{min}$. For example, in Fig. 8A we solve (1a), (17) with $\varepsilon = 0.005$ using Matlab's ODE45 subroutine. In this case, the numerical integration does not stop at $b = v_{min} \approx 0.16$ (as it should), but rather continues down to $b \approx 0.14$. Although the solutions appear to be "smooth" in the region $0.14 < b \leqslant v_{min}$, the computed values are a numerical artifact.[5] Recall that in the symmetric case, we avoided such numerical artifacts by using ODE15s, see Fig. 5B. In the asymmetric case, however, ODE15s also fails to stop at $b = v_{min} \approx 0.14$, see Fig. 8B, and rather continues down to $b \approx 0.11$.[6]

4. The boundary-value method ($n = 2$)

In Section 3 we saw that the backward-shooting method for solving the boundary-value problem (1) is unstable, especially when the number of players is large. We now present an alternative numerical method, in which (1) is solved using a boundary-value approach. For simplicity, we first present the method for the case of two players. The general case will be considered in Section 5.

In the case of two players, Eq. (1a) reduces to

$$\frac{dv_1}{db} = \frac{F_1(v_1(b))}{f_1(v_1(b))}\frac{1}{v_2(b) - b}, \qquad \frac{dv_2}{db} = \frac{F_2(v_2(b))}{f_2(v_2(b))}\frac{1}{v_1(b) - b}, \quad 0 \leqslant b \leqslant \bar{b}, \tag{19a}$$

[4] In Fig. 7, $b_1 = b_2$ and $b_3 = b_4$. Therefore, only two strategies are plotted.

[5] Indeed, the computed solution loses its monotonicity at $b = v_{min}$.

[6] The difference between the computed values of v_ε in the region $b < v_{min}$ in Fig. 8A provides further support that the computed values in these region are a numerical artifact.

with the boundary conditions

$$v_1(b=0)=v_2(0)=0, \qquad v_1(b=\bar{b})=v_2(\bar{b})=1. \tag{19b}$$

The main difficulty in solving this nonlinear boundary-value problem is due to the fact that the location $\bar{b}$ of the right boundary is unknown. In order to overcome this difficulty, we recast (19) as a function of v_2 rather than of b.[7,8] The resulting equations are

$$\begin{cases} \dfrac{dv_1}{dv_2}=\dfrac{F_1(v_1(v_2))}{f_1(v_1(v_2))}\dfrac{f_2(v_2)}{F_2(v_2)}\dfrac{v_1(v_2)-b(v_2)}{v_2-b(v_2)}, \\ \dfrac{db}{dv_2}=\dfrac{f_2(v_2)}{F_2(v_2)}[v_1(v_2)-b(v_2)], \end{cases} \qquad 0\leqslant v_2\leqslant 1, \tag{20a}$$

with the boundary conditions

$$v_1(v_2=0)=b(0)=0, \qquad v_1(v_2=1)=1. \tag{20b}$$

Unlike (19), the boundary-value problem (20) is defined on a fixed domain $v_2\in[0,1]$. The values of v_1 are given at both boundaries, whereas the value of b is only given in the left boundary.[9] This nonstandard system has a unique solution, since it is equivalent to (19) which has a unique solution (Lebrun, 1996, 1999, 2006).

4.1. Fixed-point iterations

Since (20) is a nonlinear boundary-value system, it has to be solved using a nonlinear solver, such as fixed-point iterations. We note that there are numerous ways to solve (20) using fixed-point iterations. Our specific choice is based on trial and error, and is given by

$$\begin{cases} \left[\dfrac{d}{dv_2}-\dfrac{F_1(v_1^{(k)})}{f_1(v_1^{(k)})}\dfrac{f_2(v_2)}{F_2(v_2)}\dfrac{1}{v_2-b^{(k)}}\right]v_1^{(k+1)}=-\dfrac{F_1(v_1^{(k)})}{f_1(v_1^{(k)})}\dfrac{f_2(v_2)}{F_2(v_2)}\dfrac{b^{(k)}}{v_2-b^{(k)}}, \\ \left[\dfrac{d}{dv_2}+\dfrac{v_1^{(k+1)}-b^{(k)}}{v_2-b^{(k)}}\dfrac{f_2(v_2)}{F_2(v_2)}\right]b^{(k+1)}=\dfrac{f_2(v_2)}{F_2(v_2)}\dfrac{v_1^{(k+1)}-b^{(k)}}{v_2-b^{(k)}}v_2, \end{cases} \tag{21a}$$

with the boundary conditions

$$v_1^{(k+1)}(0)=b^{(k+1)}(0)=0, \qquad v_1^{(k+1)}(1)=1, \tag{21b}$$

where $k=0,1,\ldots$ is the iteration number. In the numerical examples presented in this paper, we used the initial guess

$$v_1^{(0)}(v_2)=v_2, \qquad b^{(0)}(v_2)=\frac{v_2}{2}. \tag{22}$$

Eq. (21a) for $v_1^{(k+1)}$ is a linear first-order equation. Therefore, it is not a priori clear why its solution can satisfy the two boundary conditions in (21b). Note that a similar situation occurs in Eq. (20a) for $v_1(v_2)$. In that case, it follows from the rigorous studies of Lebrun (1996, 1999, 2006) that Eq. (19), hence also Eq. (20), has a unique solution.[10] Intuitively, this is because the right-hand side of (20a) is of the form $0/0$ at $v_2=0$, and therefore $v_1(v_2)$ has nonuniqueness at $v_2=0$. In Fibich and Gavish (2010b), we provide some informal arguments that suggest that since the right-hand side of (21a) is also of the form $0/0$ at $v_2=0$, $v_1^{(k+1)}$ is nonunique at $v_2=0$, which “enables” it to satisfy both boundary conditions.

At present, there is no proof that the iterative solution of the boundary-value method, with either the fixed-point iterations (21) or with Newton's iterations to be presented in Section 4.2, exists, or that the iterations converge. In Section 4.3 we test these two iterative approaches numerically, and observe that they converge at the expected linear and quadratic rates, respectively.

Remark 4.1. Matlab codes for the boundary-value method with the fixed-point iterations (21), and with Newton's iterations (24) are provided in the on-line supplement.

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[7] The choice of v_2 and not v_1 as the independent variable is ad-hoc. In some cases, we observed numerically that a wrong choice of the independent variable may lead to divergence (see Section 7).

[8] Amann and Leininger (1996) used the same change of variables in the analysis of all-pay auctions. Lebrun (1999) and, under a slightly different presentation, Lebrun (2009) used the same change of variables to study first-price asymmetric auctions.

[9] Note that $\bar{b}$ does not appear in (20). Its value can be recovered from the solution of (20) using the relation $\bar{b}=b(v_2=1)$.

[10] A different proof is provided by Fibich and Gavish (2010a) for the special case of power-law distributions.

4.2. Newton's iterations

We can also solve the nonlinear boundary-value problem (20) using Newton's method, which generically has a quadratic convergence rate. To do that, denote the residual as

$$R_{v_1}\big[v_1', v_1, b, v_2\big] = v_1'(v_2) - \frac{F_1(v_1)}{f_1(v_1)}\frac{f_2(v_2)}{F_2(v_2)}\frac{v_1-b}{v_2-b},$$
$$R_b\big[b', v_1, b, v_2\big] = b'(v_2) - \frac{f_2(v_2)}{F_2(v_2)}(v_1-b).$$

We would like to find a variation $\underline{\delta}(v_2) = (\delta_{v_1}(v_2), \delta_b(v_2))$ such that $(v_1+\delta_{v_1}, b+\delta_b)$ is the solution of (20), i.e.,

$$R_{v_1}\big[v_1'+\delta_{v_1}', v_1+\delta_{v_1}, b+\delta_b, v_2\big] = 0, \qquad R_b\big[b'+\delta_b', v_1+\delta_{v_1}, b+\delta_b, v_2\big] = 0.$$

Since

$$R_{v_1}\big[v_1'+\delta_{v_1}', v_1+\delta_{v_1}, b+\delta_b, v_2\big] = R_{v_1}\big[v_1', v_1, b, v_2\big] + \delta_{v_1}' + \frac{d}{dv_1}R_{v_1}\delta_{v_1} + \frac{d}{db}R_{v_1}\delta_b + \mathcal{O}\big(\|\underline{\delta}\|^2\big),$$
$$R_b\big[b'+\delta_b', v_1+\delta_{v_1}, b+\delta_b, v_2\big] \quad = R_b\big[v_1', v_1, b, v_2\big] + \delta_b' + \frac{d}{dv_1}R_b\delta_{v_1} + \frac{d}{db}R_b\delta_b + \mathcal{O}\big(\|\underline{\delta}\|^2\big),$$

the linearized equation for $\underline{\delta}$ is given by

$$\frac{d}{dv_2}\begin{pmatrix}\delta_{v_1}\\ \delta_b\end{pmatrix} + \begin{pmatrix}\frac{d}{dv_1}R_{v_1} & \frac{d}{db}R_{v_1}\\ \frac{d}{dv_1}R_b & \frac{d}{db}R_b\end{pmatrix}\begin{pmatrix}\delta_{v_1}\\ \delta_b\end{pmatrix} = -\begin{pmatrix}R_{v_1}\\ R_b\end{pmatrix}, \tag{23a}$$

where

$$\begin{cases}\dfrac{d}{dv_1}R_{v_1} = \left[\dfrac{F_1(v_1)f_1'(v_1)}{f_1^2(v_1)} - 1\right]\dfrac{f_2(v_2)}{F_2(v_2)}\dfrac{v_1-b}{v_2-b} - \dfrac{F_1(v_1)}{f_1(v_1)}\dfrac{f_2(v_2)}{F_2(v_2)}\dfrac{1}{v_2-b},\\[2mm] \dfrac{d}{db}R_{v_1} = \dfrac{F_1(v_1)}{f_1(v_1)}\dfrac{f_2(v_2)}{F_2(v_2)}\dfrac{v_1-v_2}{(v_2-b)^2},\\[2mm] \dfrac{d}{dv_1}R_b = -\dfrac{f_2(v_2)}{F_2(v_2)},\\[2mm] \dfrac{d}{db}R_b = \dfrac{f_2(v_2)}{F_2(v_2)},\end{cases} \tag{23b}$$

with the boundary conditions

$$\delta_{v_1}(0) = \delta_b(0) = 0, \qquad \delta_{v_1}(1) = 0. \tag{23c}$$

Therefore, Newton's iterations are

$$v_1^{(k+1)} = v_1^{(k)} + \delta_{v_1}^{(k)}, \qquad b^{(k+1)} = b^{(k)} + \delta_b^{(k)}, \quad k = 0, 1, 2, \ldots, \tag{24}$$

where $\underline{\delta}^{(k)} = (\delta_{v_1}^{(k)}, \delta_b^{(k)})$ is the solution of (23) where $R_{v_1}^{(k)} = R_{v_1}[(v_1')^{(k)}, v_1^{(k)}, b^{(k)}, v_2]$, and $R_b^{(k)} = R_b[(v_1')^{(k)}, v_1^{(k)}, b^{(k)}, v_2]$. Note that the boundary-conditions (23c) were determined such that $(v_1+\delta_{v_1}, b+\delta_b)$ satisfies the boundary-conditions (20b).

As in the case of the fixed-point iterations (21), since Eq. (23a) is a system of two linear first-order equations, it is not clear why its solution can satisfy the three boundary conditions (23c). In Fibich and Gavish (2010b), we provide some informal arguments that suggest that the nonuniqueness at $v_2 = 0$ "enables" the solution to satisfy the three boundary conditions (23c).

4.3. Simulations

We solve the boundary-value problem (19) with $F_1(v) = v$ and $F_2(v) = v^2$. In this case, the maximal bid $\bar{b}$ can be calculated analytically (Marshall et al., 1994) and is given by $\bar{b} = 37/64$. Hence, we can monitor the error $E^{(k)} = |b^{(k)}(1) - \bar{b}|$, as a function of the iteration number k.

We first solve this problem using the boundary-value formulation (20) with the fixed-point iterations (21). As expected, the error $E^{(k)}$ decreases linearly, going down from $E^{(0)} \approx 0.1$ to $E^{(25)} \approx 8.5 \times 10^{-10}$ in 25 iterations, see Fig. 9A. We then solve the same problem using the boundary-value formulation (25) with Newton's method (24). As expected, the observed convergence rate is quadratic, going down from $E^{(0)} \approx 0.1$ to $E^{(5)} \approx 8.5 \cdot 10^{-10}$ in 5 iterations, see Fig. 9A.

In both cases, the limiting error stabilizes at $E^{(\infty)} := \lim_{k\to\infty} E^{(k)} \approx 8.5 \times 10^{-10}$. The limiting error does not go down to zero, because of the discretization error of the finite-difference approximation of the differentiation operator $\frac{d}{dv_2}$ in (20). In

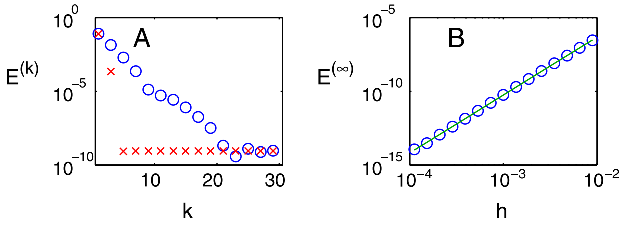

Fig. 9. A: The error $E^{(k)} = |b^{(k)}(1) - \bar{b}|$ as function of the iteration number k, for the boundary-value method with the fixed-point iterations (21) (∘) and with Newton's iterations (24) (×). B: The limiting error $E^{(\infty)} = \lim_{k\to\infty} E^{(k)}$ as a function of the mesh size h (∘). Solid line is the fitted curve $E^{(\infty)} \sim 36.95h^{3.94}$.

other words, the numerical solution converges to the solution of the discrete approximation of (20), which is not identical to the (continuous) solution of (20). Indeed, $E^{(\infty)}$ is the same for both the fixed-point iterations (21) and for Newton's method (24), since we used the same discretization scheme in both cases. Moreover, in the simulations of Fig. 9A we used a uniform grid with mesh-size h and a fourth-order finite-difference approximation of $\frac{d}{dv_2}$. Therefore $E^{(\infty)}$ should scale as h^4, as we indeed confirm in Fig. 9B.

5. The boundary-value method (general n)

In Section 4 we presented a numerical method for solving the system (1) with 2 players using a boundary-value formulation. We now generalize this approach for any number of players. As in the two-player case, in order to solve this problem on the fixed interval [0, 1], we recast (1) as a function of v_n rather than of b. The resulting equations are given by

$$\begin{cases} \dfrac{dv_i}{dv_n} = \dfrac{F_i(v_i) f_n(v_n)}{f_i(v_i) F_n(v_n)} \dfrac{\left(\frac{1}{n-1}\sum_{j=1}^{n}\frac{1}{v_j-b}\right) - \frac{1}{v_i-b}}{\left(\frac{1}{n-1}\sum_{j=1}^{n}\frac{1}{v_j-b}\right) - \frac{1}{v_n-b}}, \quad i = 1,\ldots,n-1, \\ \dfrac{db}{dv_n} = \dfrac{f_n(v_n)}{F_n(v_n)} \dfrac{1}{\left(\frac{1}{n-1}\sum_{j=1}^{n}\frac{1}{v_j-b}\right) - \frac{1}{v_n-b}}. \end{cases} \tag{25a}$$

These equations are solved on the fixed domain $0 \leqslant v_n \leqslant 1$, subject to the boundary conditions

$$v_i(0) = b(0) = 0, \qquad v_i(1) = 1, \quad i = 1,\ldots,n-1. \tag{25b}$$

In order to generalize the fixed-point iterations (21) to n-players, we first simplify (25a) by multiplying the denominator and enumerator of each equation by $(n-1)\prod_{l=1}^{n}(v_l - b)$ to obtain

$$\begin{cases} \dfrac{dv_i}{dv_n} = \dfrac{F_i(v_i) f_n(v_n)}{f_i(v_i) F_n(v_n)} \dfrac{\sum_{j=1}^{n}\left[\prod_{l\neq j}(v_l-b)\right] - (n-1)\prod_{l\neq i}(v_l-b)}{\sum_{j=1}^{n}\left[\prod_{l\neq j}(v_l-b)\right] - (n-1)\prod_{l\neq n}(v_l-b)}, \quad i = 1,\ldots,n-1, \\ \dfrac{db}{dv_n} = \dfrac{f_n(v_n)}{F_n(v_n)} \dfrac{(n-1)\prod_{l=1}^{n}(v_l-b)}{\sum_{j=1}^{n}\left[\prod_{l\neq j}(v_l-b)\right] - (n-1)\prod_{l\neq n}(v_l-b)}. \end{cases} \tag{26}$$

The fixed-point iterations that generalize (21) are

$$\begin{cases} \dfrac{dv_i^{(k+1)}}{dv_n} = \dfrac{F_i(v_i^{(k)}) f_n(v_n)}{f_i(v_i^{(k)}) F_n(v_n)} \\ \quad \times \dfrac{\sum_{j=1}^{n}\left[\prod_{l\leqslant i, l\neq j}(v_l^{(k+1)} - b^{(k)})\prod_{l>i, l\neq j}(v_l^{(k)} - b^{(k)})\right] - (n-1)\prod_{l<i}(v_l^{(k+1)} - b^{(k)})\prod_{l>i}(v_l^{(k)} - b^{(k)})}{\sum_{j=1}^{n}\left[\prod_{l<i, l\neq j}(v_l^{(k+1)} - b^{(k)})\prod_{l\geqslant i, l\neq j}(v_l^{(k)} - b^{(k)})\right] - (n-1)\prod_{l<i}(v_l^{(k+1)} - b^{(k)})\prod_{l>i}(v_l^{(k)} - b^{(k)})}, \\ i = 1,\ldots,n-1, \\ \dfrac{db}{dv_n} = \dfrac{f_n(v_n)}{F_n(v_n)} \dfrac{(n-1)(v_n - b^{(k+1)})\prod_{l\neq n}(v_l^{(k+1)} - b^{(k)})}{\sum_{j=1}^{n}\left[\prod_{l\neq j}(v_l^{(k+1)} - b^{(k)})\right] - (n-1)\prod_{l\neq n}(v_l^{(k+1)} - b^{(k)})}, \end{cases} \tag{27a}$$

with the boundary conditions

$$v_i^{(k+1)}(0) = b^{(k+1)}(0) = 0, \qquad v_i^{(k+1)}(1) = 1. \tag{27b}$$

Newton's method that generalizes (24) is

$$\delta' + \mathbf{J}\delta = -\mathbf{R}, \tag{28a}$$

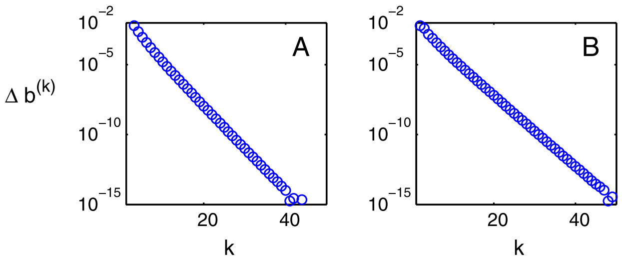

Fig. 10. Difference $\Delta b^{(k)} = |\bar{b}^{(k)} - \bar{b}^{(k-1)}|$ as a function of the iteration number k (∘), for the boundary-value method (25) with the fixed-point iterations (27). Simulations of A: Fig. 11, and B: Fig. 12 (see below).

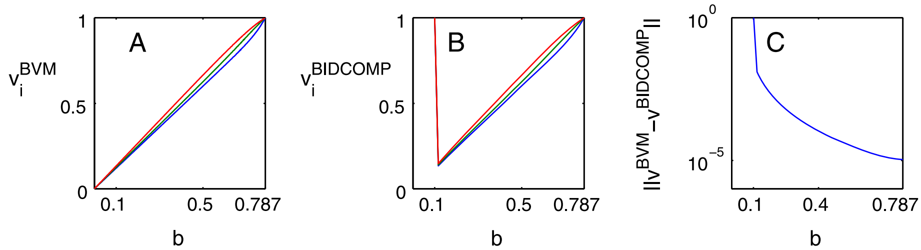

Fig. 11. Solution of (1) with three players with distribution $F_i = \{v, v^2, v^3\}$ computed with A: the boundary-value method (25) with the fixed-point iterations (27), and B: backward shooting (BIDCOMP^2). C: The difference $\|\mathbf{v}^{BVM}(b) - \mathbf{v}^{BIDCOMP}(b)\|$.

where $\mathbf{J}$ is the Jacobian matrix

$$\mathbf{J} = \begin{bmatrix} \frac{\partial}{\partial v_1} R_{v_1} & \frac{\partial}{\partial v_2} R_{v_1} & \cdots & \frac{\partial}{\partial v_{n-1}} R_{v_1} & \frac{\partial}{\partial b} R_{v_1} \\ \vdots & & & & \vdots \\ \frac{\partial}{\partial v_1} R_{v_{n-1}} & \frac{\partial}{\partial v_2} R_{v_{n-1}} & \cdots & \frac{\partial}{\partial v_{n-1}} R_{v_{n-1}} & \frac{\partial}{\partial b} R_{v_{n-1}} \\ \frac{\partial}{\partial v_1} R_b & \frac{\partial}{\partial v_2} R_b & \cdots & \frac{\partial}{\partial v_{n-1}} R_b & \frac{\partial}{\partial b} R_b \end{bmatrix}, \tag{28b}$$

$\mathbf{R} = (R_{v_1}, R_{v_2}, \ldots, R_{v_{n-1}}, R_b)^T$ is the residual vector, and

$$R_{v_i} = \frac{dv_i}{dv_n} - \frac{F_i(v_i) f_n(v_n)}{f_i(v_i) F_n(v_n)} \frac{\left(\frac{1}{n-1}\sum_{j=1}^n \frac{1}{v_j - b}\right) - \frac{1}{v_i - b}}{\left(\frac{1}{n-1}\sum_{j=1}^n \frac{1}{(v_j - b)}\right) - \frac{1}{v_n - b}},$$

$$R_b = \frac{db}{dv_n} - \frac{f_n(v_n)}{F_n(v_n)} \frac{1}{\left(\frac{1}{n-1}\sum_{j=1}^n \frac{1}{v_j - b}\right) - \frac{1}{v_n - b}}. \tag{28c}$$

These equations are solved with the boundary conditions

$$\delta(0) = 0, \qquad \delta_{v_i}(1) = 0, \quad i = 1, \ldots, n-1. \tag{28d}$$

As in the case of two types of players, we do not provide a proof for the existence of the iterative solutions, or for their convergence. In Section 5.1 we test the Newton's iterations numerically, and observe that they converge in a quadratic rate.

5.1. Simulations

We first solve Eq. (1) with $n = 3$ players with distributions $F_i = \{v, v^2, v^3\}$, using the boundary-value formulation (25) and the fixed-point iterations (27). Since an explicit expression for $\bar{b}$ is not available in this case, we monitor the value of $\Delta b^{(k)} = |b^{(k)}(1) - b^{(k-1)}(1)|$. Fig. 10A shows that the convergence rate is linear, and that $\Delta b^{(k)}$ goes down from $\Delta b^{(0)} \approx 10^{-2}$ to $\Delta b^{(40)} \approx 10^{-15}$ in ≈ 40 iterations.

In order to confirm that the fixed-point iterations converged to the unique solution of (1), we compare the solution $\mathbf{v}^{BVM} = (v_1^{BVM}, \ldots, v_n^{BVM})$ that the fixed-point iterations (27) converged to, see Fig. 11A, with the solution $\mathbf{v}^{BIDCOMP} = (v_1^{BIDCOMP}, \ldots, v_n^{BIDCOMP})$ computed using the freely available BIDCOMP^2 backward-shooting program, see Fig. 11B. In this case, BIDCOMP^2 calculates the solution only down to $b = b_{min} \approx 0.1$. At $b_{min} \approx 0.1$ the BIDCOMP^2 solution breaks down, and sharply increases towards $v = 1$, demonstrating the inherent numerical instability of type-II backward solutions beyond v_{min}, see Section 3.4. In contrast, the boundary-value method does not suffer from this instability. Rather, the solution $\mathbf{v}^{BVM}$ is computed for $0 \leqslant b \leqslant \bar{b}$, and in particular satisfies the left boundary condition (1b), see Fig. 11A.

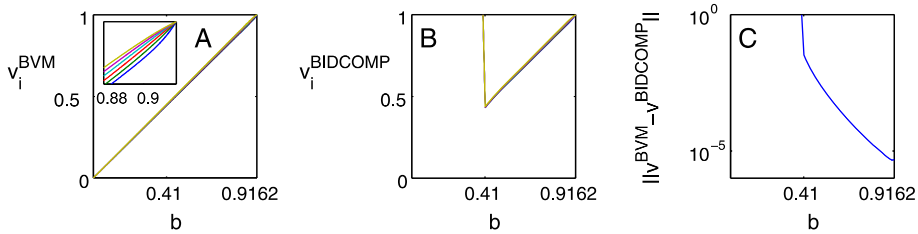

Fig. 12. Same as Fig. 11 with $n = 6$ and the distributions $F_i = \{v, v^{3/2}, v^2, v^{5/2}, v^3, v^{7/2}\}$. Inset in A is a zoom-in on the region $0.88 \leqslant b \leqslant \bar{b}$.

The difference in the value of $\bar{b}$ computed by the two methods is $|\bar{b}^{BVM} - \bar{b}^{BIDCOMP}| = 1.08 \cdot 10^{-5}$, confirming that the two methods converged to the same solution of (1). As b decreases from $\bar{b}$, the difference between the two solutions

$$\left\|\mathbf{v}^{BVM}(b) - \mathbf{v}^{BIDCOMP}(b)\right\| = \sqrt{\sum_{i=1}^{n}\left(v_i^{BVM}(b) - v_i^{BIDCOMP}(b)\right)^2} \tag{29}$$

increases from $1.08 \cdot 10^{-5}$ at $b = \bar{b} \approx 0.787$ to 0.012 at $b = v_{min} + 0.01 \approx 0.11$. Then at $v_{min} \approx 0.1$ it jumps to 0.9, see Fig. 11C. We attribute the increase in the difference (29) in the region $v_{min} < b < 1$ to the growth of the analytic error of backward as b decreases from $\bar{b}$, and the sudden jump in the difference (29) at $b = v_{min}$ to the numerical error of type-II backward solutions, see Section 3.4.

We next solve Eq. (1a) with 6 players with distributions $F_i = \{v, v^{3/2}, v^2, v^{5/2}, v^3, v^{7/2}\}$, using the boundary-value method (25) with the fixed-point iterations (27). As in the case of three players, the convergence rate is linear, and $\Delta b^{(k)}$ goes down by 13 orders of magnitude in ≈ 50 iterations, see Fig. 10B.

In Fig. 12A we plot the solution $\mathbf{v}^{BVM}$ that the iterations (27) converged to. Although the six curves $\{v_i^{BVM}\}_{i=1}^{6}$ are nearly indistinguishable, they are different and well resolved, see inset graph of Fig. 12A. In this case, BIDCOMP[2] calculates the solution only down to $b = v_{min} \approx 0.41$, see Fig. 12B, at which point it breaks down, and sharply increases towards $v = 1$. As expected, with six players, v_{min} is significantly larger with three players, see Fig. 11B, confirming that the analytic instability of the backward solutions becomes more severe as the number of players increases. In contrast, the solution $\mathbf{v}^{BVM}$ is computed for $0 \leqslant b \leqslant \bar{b}$, and in particular satisfies the boundary condition (1b), see Fig. 12A.

As in the three-player case, the difference in the values of $\bar{b}$ computed by the two methods is $|\bar{b}^{BVM} - \bar{b}^{BIDCOMP}| = 4.6 \times 10^{-6}$, showing that the two methods converged to the same solution of (1). As b decreases from $\bar{b}$, the difference between the solutions $\|\mathbf{v}^{BVM}(b) - \mathbf{v}^{BIDCOMP}(b)\|$ increases from 4.6×10^{-6} at $b = \bar{b} \approx 0.9162$ to 0.033 at $b = v_{min} + 0.01 \approx 0.42$. Then at $v_{min} \approx 0.41$ it jumps to 0.6, see Fig. 12C. As before, we attribute the increase in the difference (29) in the region $v_{min} < b < 1$ to the growth of the analytic error of backward solutions as b decreases from $\bar{b}$, and the jump in the difference (29) at $b = v_{min}$ to the numerical error of type-II backward solutions.

The convergence of the fixed-point iterations (27) does not slow down as the number of players increases. Indeed, both simulations converged (i.e., $\Delta b^{(k)}$ went down by 10^{-13}) in 40–50 iterations, see Fig. 10. Moreover, the simulations in Section 6.1 with up to 450 players also converged in 25–50 iterations. *Therefore, unlike backward shooting, the boundary-value method does not become less efficient as the number of players increases.*

6. Applications

In this section we apply the boundary-value method to several problems that could not been solved using backward shooting.

6.1. Large auctions

In Section 3 we saw that the instability of backward solutions becomes more severe as the number of players increases. Therefore, the study numerically large asymmetric first-price auctions with backward shooting is limited. We now use the boundary-value method to study large asymmetric first-price auctions, namely we solve Eq. (1) using the boundary-value formulation (25) with the fixed-point iterations (27).

We begin with the case of $n = 9$ players with the nine distributions

$$\begin{aligned}
&F_k = c_k \cdot \operatorname{erf}\left(\frac{\alpha_k}{\alpha_k + 1}\frac{v}{\sqrt{2}}\right), \quad \alpha_k = 1, 2, 3 \quad \text{(truncated normal distribution)}\\
&F_k = c_k \cdot v^{\beta_k}, \quad \beta_k = 1, 2, 3 \quad \text{(power)}\\
&F_k = c_k \cdot \left[1 - \exp\left(\frac{v}{\gamma_k}\right)\right], \quad \gamma_k = 1, 2, 3 \quad \text{(Weibull)},
\end{aligned} \tag{30}$$

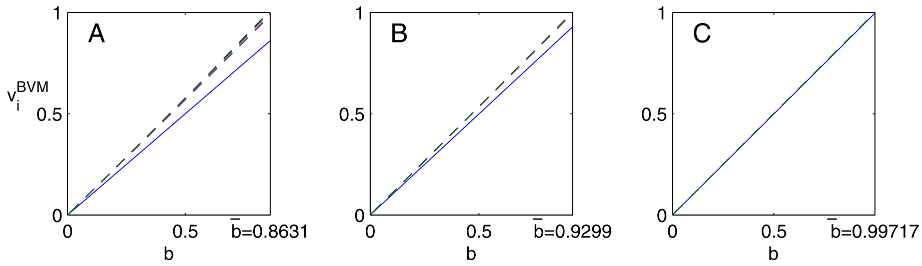

Fig. 13. Solution of (1) with n players (dashed curves) divided into nine equal groups. The players of the ith group draw their value according to the distribution F_i from (30). Solid line is $v = b$. A: $n = 9$, B: $n = 18$, C: $n = 450$.

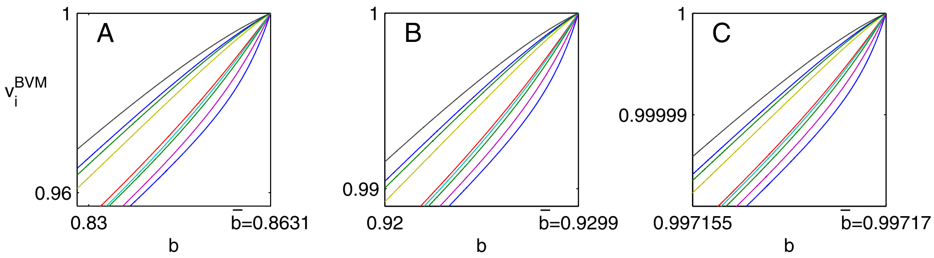

Fig. 14. Zoom-in on the data of Fig. 13 for b near $\bar{b}$ and $v_i(b)$ near 1.

where $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, and $\{c_k\}$ are set such that $F_k(1) = 1$. The nine curves $\{v_i(b)\}_{i=1}^{9}$ are nearly indistinguishable, see Fig. 13A. In order to show that these nine curves are different and well resolved, we plot them near $b = \bar{b}$ and $v_i = 1$, see Fig. 14A.

Next, we solve Eq. (1) with $n = 18$ players, when the players are divided into nine groups, each group has two players, and the players of the ith group draw their value according to the distributions $\{F_1, F_2, \dots, F_9\}$ given by (30). In this case, $b_1(v) = b_2(v)$, $b_3(v) = b_4(v), \dots, b_{17}(v) = b_{18}(v)$. Therefore, only nine curves are plotted in Fig. 13B. The nine curves in Fig. 13B are closer to each other and to $v = b$ than in the $n = 9$ case in Fig. 13A, suggesting that

$$\lim_{n\to\infty} b_i(v) = v, \quad 1 \leqslant i \leqslant n. \tag{31}$$

These nine curves are different and well resolved, see Fig. 14B.

Finally, we solve Eq. (1) with $n = 450$ players which are divided into nine groups of 50 players, and the players of the ith group draw their value according to the distribution F_i given by (30). The nine curves are indistinguishable from each other and from the curve $v = b$, see Fig. 13C, in agreement with (31). These nine curves are different and well resolved, see Fig. 14C.

6.2. Asymptotic revenue equivalence

One of the most fundamental results in auction theory is the Revenue Equivalence Theorem, which states that the expected revenue of the seller in equilibrium is independent of the auction mechanism, so long as the equilibrium has the properties that the buyer with the highest value wins, and any buyer with the lowest possible value has zero expected surplus. This condition holds for symmetric auctions, but not for asymmetric auctions. Hence, asymmetric auctions are generally not revenue equivalent (see, e.g., Maskin and Riley, 2000). Nevertheless, since asymmetric auctions "become symmetric" as $n \to \infty$, see Section 6.1, asymmetric auctions become *revenue equivalent* as $n \to \infty$ (Bali and Jackson, 2002).

We now use the boundary-value method to compute *the rate* at which large asymmetric auctions become *revenue equivalent*. Consider two groups of players, $n/2$ players with distribution $F_1(v)$, and $n/2$ players with distribution $F_2(v)$. Since the bidding strategy of all bidders within the same group is the same, the system (1) reduces to

$$\begin{aligned} v_1'(b) &= \frac{F_1(v_1(b))}{f_1(v_1(b))} \frac{1}{2(n-1)} \left[\frac{n}{v_2(b) - b} - \frac{n-2}{v_1(b) - b} \right], \\ v_2'(b) &= \frac{F_2(v_2(b))}{f_2(v_2(b))} \frac{1}{2(n-1)} \left[\frac{n}{v_1(b) - b} - \frac{n-2}{v_2(b) - b} \right], \end{aligned} \tag{32a}$$

with the boundary conditions

$$v_1(0) = v_2(0) = 0, \qquad v_1(\bar{b}) = v_2(\bar{b}) = 1. \tag{32b}$$

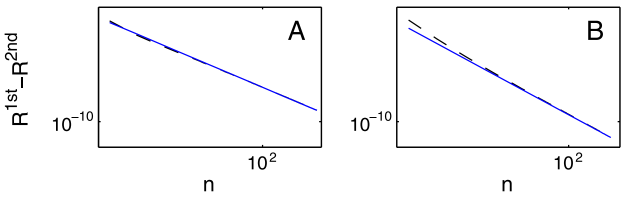

Fig. 15. Revenue difference between first- and second-price auctions when the players are divided into two groups, each with $n/2$ players (dashed curve). Data is plotted on a logarithmic scale. A: F_1 and F_2 are given by (33). Solid curve is the numerical fitted curve $0.04n^{-3}$. B: $F_1 = v$ and $F_2 = v^2$. Solid curve is the numerical fitted curve $0.035n^{-4}$.

We solve (32) using the boundary-value method (25) with Newton's iterations (28), for $n = 2, 4, \ldots, 400$ players. Then, we calculate the seller's expected revenue, which is given by the one-dimensional integral $R^{1\text{st}}(n) = \bar{b} - \int_0^{\bar{b}} \prod_{i=1}^{n} F_i(v_i(b))\,db$.[11] In the case of an asymmetric second-price auction, the seller's expected revenue is given explicitly by $R^{2\text{nd}}(n) = 1 - \int_0^1 [(1-n)\prod_{i=0}^{n} F_i(v) + \sum_{i=1}^{n}\prod_{j\neq i} F_j(v)]\,dv$.

In Fig. 15A we plot the revenue difference between first- and second-price asymmetric auctions as a function of n for the case of two groups, each with $n/2$ players, with the distributions

$$F_1 = \left[1 - \exp\left(\frac{v}{2}\right)\right] \Big/ \left[1 - \exp\left(\frac{1}{2}\right)\right] \quad \text{(Weibull)}, \qquad F_2 = \sqrt{v}. \tag{33}$$

In this case, the revenue difference behaves as

$$R^{1\text{st}}(n) - R^{2\text{nd}}(n) \sim \frac{0.04}{n^3}. \tag{34}$$

In Fig. 15B we repeat this procedure with $F_1 = v$ and $F_2 = v^2$, for $n = 2, 4, \ldots, 280$ players. In this case, the revenue difference behaves as

$$R^{1\text{st}}(n) - R^{2\text{nd}}(n) \sim \frac{0.035}{n^4}. \tag{35}$$

The most noticeable feature in Fig. 15 is that the revenue differences between first-price auctions and second-price auctions decay very rapidly with the number of players. For example, already with 6 players the revenue differences are 0.02% or less. The rate at which the revenue differences go to zero is not the same in the two examples. Nevertheless, in these two examples, as well as in additional examples that we checked (data not shown), the decay rate of the revenue difference was always at least $\mathcal{O}(1/n^3)$. This numerical observation is relevant to the open problem of revenue ranking of first and second asymmetric auctions, as it suggests that:

1. In the case of large auctions, the revenue differences are so small that the problem of revenue ranking is more of academic interest than of practical value.
2. Already with six players, the revenue differences are typically in the fourth or fifth digit.

6.3. Relaxing the assumption of stochastic dominance

Kirkegaard (2009) noted that the standard assumption in the existing analytic studies of asymmetric first-price auctions is that the distributions $\{F_i\}$ can be ordered according to first-order conditional stochastic dominance, i.e.,

$$F_1 \succ F_2 \succ \cdots \succ F_n, \tag{36}$$

where $F_i \succ F_j$ if $d(F_i(v)/F_j(v))/dv < 0$. Because $F_k(1) = 1$ for $k = 1, \ldots, n$, relation (36) implies that

$$F_1(v) > F_2(v) > \cdots > F_n(v), \quad 0 < v < 1.$$

Therefore, the distribution $\{F_i\}$ do not cross each other in $(0, 1)$.

A natural question is whether the analytic results proved under the assumption of stochastic dominance remain true when this assumption is removed. For example, Maskin and Riley (2000) proved the following result:

Lemma 6.1. *Consider a first-price auction with two players with distributions F_1 and F_2. If $F_1 \prec F_2$, then $\mu_1 \leqslant \bar{b} \leqslant \mu_2$, where*

[11] In this approach one can compute the expected revenue much more efficiently and accurately than with the commonly used Monte Carlo approach (Fibich and Gavious, 2003).

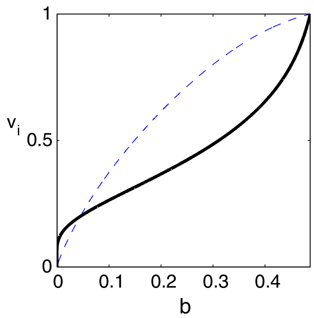

Fig. 16. Solution of (1) with the distributions given by (38), calculated using the boundary-value method with the fixed-point iterations (21).

$$\mu_i = \int_0^1 v f_i(v)\,dv, \tag{37}$$

is bidder's i expected value.

We now use the boundary-value method to show that Lemma 6.1 does not hold when F_1 and F_2 cannot be ordered according to first-order conditional stochastic dominance. To do that, we solve (1) for two players with

$$F_1(v) = \operatorname{erf}\left(\frac{v}{2\sqrt{2}}\right) \Big/ \operatorname{erf}\left(\frac{1}{2\sqrt{2}}\right), \qquad F_2(v) = F_1(v) + 3v(1-v)\left(\frac{1}{2} - v\right), \tag{38}$$

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. Since F_1 and F_2 cross at $v = 1/2$, they cannot be ordered according to first-order conditional stochastic dominance. Moreover, for the distributions (38), $\mu_1 = \mu_2 \approx 0.36$. Therefore, if Lemma 6.1 remains valid without the stochastic dominance assumption, then $\bar{b} = \mu_1 \approx 0.36$. Our simulation shows that $\bar{b} \approx 0.48$, see Fig. 16.[12] Hence, we conclude that Lemma 6.1 does not hold in the general case.

Kirkegaard (2009) stated that "*all* existing classes of numerical examples share the common feature that the number of times the bid strategies cross is at most one". Therefore, although "it is straightforward to construct examples with several crossings, [...] with the current specifications, the numerical literature will be unable to provide examples with several crossings".

We now use the boundary-value method to provide a numerical example of bidding strategies with more than one crossing. Specifically, we solve Eq. (1) with three players with the distributions

$$\begin{aligned} &F_1 = v, \qquad F_2 = v + 2v^2(1-v^2)(0.25-v^2)(0.75-v^2), \\ &F_3 = v - 3v^2(1-v^2)(0.25-v^2)(0.75-v^2). \end{aligned} \tag{39}$$

These distributions cross each other twice in the interval $(0, 1)$ at $v = \sqrt{0.25}$ and $v = \sqrt{0.75}$, see Fig. 17A. Therefore, by Kirkegaard (2009, Proposition 3), the bidding strategies in this case should cross each other twice in the interval $(0, 1)$, as indeed can be observed in Fig. 17B.

7. Discussion

In this study, we showed that the standard backward-shooting method for computing the equilibrium strategies of asymmetric first-price auctions is inherently unstable. This instability becomes more severe as the number of players increases, and cannot be eliminated by changing the numerical methodology of the backward solver. Then, we proposed a novel boundary-value method for computing the equilibrium strategies of asymmetric first-price auctions. Unlike backward shooting, the boundary-value method is stable.

Our simulations show that the boundary-value method is considerably more robust than the backward-shooting method. In particular, it can compute the equilibrium strategies even when the number of players is large. The boundary-value method is also not restricted to certain distributions and it can accommodate distributions of mixed types. In contrast, the current backward-shooting programs are restricted to several common types of distributions, and do not allow distributions with mixed types.

A key element in the boundary-value method is the change of the independent variable from b to v_n, which fixes the domain over which the system of ordinary differential equations is solved. Our simulations show that the convergence of

[12] At first glance, it seems that special numerical treatment is required to handle the sharp variation of $v_2(b)$ near $b = 0$, see Fig. 16. However, in Fig. 16 we solve Eq. (20b) numerically for $v_1(v_2)$ and $b(v_2)$, which do not have an infinite derivative at the left boundary.

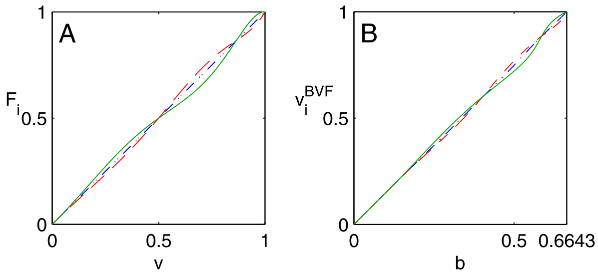

Fig. 17. A: Distributions F_1 (dash-dotted), F_2 (dashed) and F_3 (solid) given by (39). B: Solution of (1) with the distributions (39).

the fixed-point iterations can depend on the choice of the independent variable. For example, in Section 4.3 we computed the equilibrium strategies with $F_1(v) = v$ and $F_2(v) = v^2$ by changing the independent variable from b to v_2 and solving the transformed nonlinear system using the fixed-point iterations. The same fixed-point iterations, however, would diverge if we choose v_1 as the independent variable. Further research is needed in order to eliminate the ad-hoc choice of the independent variable.

In the boundary-value method, the transformed nonlinear boundary-value problem is solved using either fixed-point or Newton iterations. There is a trade-off between efficiency and ease of implementation between these two methods: The fixed-point iterations are simpler to implement (especially in the case of many different players), but converge slower than Newton's iterations. Therefore, we chose to use the fixed-point iterations for auctions with more than two different players, and Newton's iterations for auctions with two large groups of players. Performance and implementation details are further discussed in Fibich and Gavish (2010b).

In this work we did not study the sensitivity of the fixed-point iterations and Newton's iterations to the initial guess. However, the fact that all the simulations in this study converged with the same initial guess, see (22), strongly suggests that both the fixed-point iterations and Newton's iterations are insensitive to the initial guess.

In this study, we showed analytically and numerically that the backward-shooting method becomes unstable as the number of players increases. In contrast, no such problem arises with the boundary-value method. In addition, the convergence rate of the fixed-point iterations with 9–450 players is the same as with 3–6 players. This shows that the boundary-value method does not become less efficient as the number of players increases. Therefore, in contrast to the backward shooting method, the boundary-value method is well suited for studying large auctions.

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Supplementary material

The online version of this article contains additional supplementary material.
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